

## RATIONAL POINTS ON CURVES OVER FUNCTION FIELDS

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ABSTRACT. We provide in this paper an upper bound for the number of rational points on a curve defined over a one variable function field over a finite field. The bound only depends on the curve and the field, but not on the Jacobian variety of the curve.

Let  $k$  be a finite field of positive characteristic  $p$  and  $\mathcal{C}$  a smooth, projective, geometrically connected curve defined over  $k$  of genus  $g$ . Denote by  $K = k(\mathcal{C})$  its function field. Let  $K_s$  be a separable closure of  $K$ . Given a smooth, projective, geometrically connected curve  $X$  defined over  $K$  of genus  $d \geq 2$ , the analogue of the Mordell's conjecture asks whether the set  $X(K)$  is finite.

This does not come without a constraint, otherwise this question would have a trivial negative answer. One has to assume that  $X$  is non-isotrivial. This means that there does not exist a smooth projective geometrically connected curve  $X_0$  defined over a finite extension  $l$  of  $k$  and a common extension  $L$  of both  $K$  and  $l$  such that  $X \times_K L \cong X_0 \times_l L$  (cf. [Sa66]). Under the aforementioned condition the finiteness of  $X(K)$  is a theorem due to Samuel [Sa66].

Our purpose in this note is to give an effective bound for the cardinality of the set  $X(K)$  in terms of the minimal number of invariants associated with our given geometric situation. Namely, our upper bound will depend just on  $d$ ,  $g$  and the conductor of  $X$  (this will be defined later in the text). Let us insist on the fact that this bound does not depend on the Jacobian of the curve  $X$ , in particular there is no need of calculating the  $K$ -rank of the Jacobian to effectively use the bound.

The history of explicit upper bounds for  $\#X(K)$  starts with the work of Szpiro [Sz81] which in fact gives an explicit upper bound for the height of points in  $X(K)$ . This however depends on the geometry of a semi-stable fibration on curves  $\phi : \mathcal{X} \rightarrow \mathcal{C}$  which gives a minimal model of  $X/K$  over  $\mathcal{C}$ .

We will follow another approach. This is based on a work of Buïum and Voloch [BuVo96] that gives an explicit bound for a conjecture of Lang. This conjecture has Mordell's conjecture as a special case. Another very important ingredient is an inequality relating the conductor of the curve and the conductor of its Jacobian  $J$  which originates in [Bl87] (see Proposition 7). Denote by  $j : X \hookrightarrow J$  an embedding of  $X$  into  $J$ .

In the next theorem the authors need the hypothesis:  $X$  is defined over  $K$  but not over  $K^p$ . It turns out that this is equivalent to the Kodaira-Spencer map of the

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Jacobian variety  $J$  of  $X$  being non-zero (for more details see [Vo91] and [Sz81]). This hypothesis also implies that  $X/K$  is non-isotrivial.

**Theorem 1** (Buium-Voloch). [BuVo96, Theorem] *Let  $k$  be a finite field of characteristic  $p$ ,  $K$  a one variable function field over  $k$ ,  $X/K$  a smooth, projective, geometrically curve defined over  $K$  of genus  $d \geq 2$ . We suppose that  $X$  is not defined over  $K^p$ . Let  $\Gamma$  a subgroup of  $J(K_s)$  such that  $\Gamma/p\Gamma$  is finite. The following inequality holds :*

$$\#(X \cap \Gamma) \leq \#(\Gamma/p\Gamma) \cdot p^d \cdot 3^d \cdot (8d - 2) \cdot d!$$

**Remark 2.** Take  $\Gamma = J(K)$ . Then by the Mordell-Weil theorem  $J(K)/pJ(K)$  is a finite group. Writing  $J(K) = \mathbb{Z}^r \times J(K)_{\text{tor}}$  where  $r = \text{rk}(J(K))$ , one has  $J(K)/pJ(K) = (\mathbb{Z}/p\mathbb{Z})^r \times J(K)_{\text{tor}}/pJ(K)_{\text{tor}}$ . Its order is bounded from above by  $p^{d+r}$ . Next we discuss an upper bound for the rank.

**Remark 3.** Let  $k$  be any field and  $\mathcal{C}$  smooth projective geometrically connected curve over  $k$ . Denote by  $K = k(\mathcal{C})$  its function field. Let  $A/K$  be a non-constant abelian variety over  $K$  and denote by  $(\tau, B)$  its  $K/k$ -trace (cf. [La83]). Let  $\bar{k}$  be an algebraic closure of  $k$ . A theorem due to Lang and Néron ([La83], [LaNe59]) states that the quotient group  $A(\bar{k}(\mathcal{C}))/\tau B(\bar{k})$  is a finitely generated abelian group. A fortiori, the quotient group  $A(K)/\tau B(k)$  is also finitely generated. Ogg in the 60's (cf. [Ogg62], [Ogg67]) produced the following upper bound for the rank of the geometric quotient  $A(\bar{k}(\mathcal{C}))/\tau B(\bar{k})$  (hence of  $A(K)/\tau B(k)$ ). Below we define the conductor  $f_{A/K}$  of  $A/K$ . Let  $d_0 = \dim(B)$ . Then the upper bound is  $2d(2g - 2) + f_{A/K} + 4d_0 \leq 4dg + f_{A/K}$ . In particular, if  $K$  is a one variable function field over a finite field, then  $\text{rk}(A(K)) \leq 4dg + f_{A/K}$ .

**Definition 4.** Let  $\ell$  be a prime number different from the characteristic of  $k$ . Denote by  $T_\ell(A)$  the  $\ell$ -adic TATE module of  $A$  and define  $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ . For each place  $v$  of  $K$  denote by  $I_v$  an inertia group at  $v$  (well-defined up to conjugation). Let  $\epsilon_v$  be the codimension of the subgroup of  $I_v$ -invariants  $V_\ell(A)^{I_v}$  in  $V_\ell(A)$ . Let  $\delta_v$  be the Swan conductor of  $H_{\text{ét}}^1(A_{K_s}, \mathbb{Q}_\ell)$  (cf. [Se69]). Define the conductor divisor  $\mathfrak{F}_{A/K} = \sum_v (\epsilon_v + \delta_v) \cdot v$ . Its degree is denoted by  $f_{A/K}$ .

**Definition 5.** A model of  $X/K$  over  $\mathcal{C}$  is a smooth projective geometrically connected surface  $\mathcal{X}$  defined over  $k$  and a proper flat morphism  $\phi : \mathcal{X} \rightarrow \mathcal{C}$ . Each place  $v$  of  $K$  is identified with a point of  $\mathcal{C}$ . Denote by  $\kappa_v$  the residue field at  $v$  (which is a finite field) and let  $\bar{\kappa}_v$  be an algebraic closure of  $\kappa_v$ . Denote by  $\mathcal{X}_v$  the fiber of  $\phi$  at  $v$ . For an algebraic variety  $Z$  defined over a field  $l$  and for an extension  $L$  of  $l$  denote  $Z_L = Z \times_l L$ .

**Definition 6.** Fix a place  $v$  of  $K$ . The Artin conductor of the curve  $X$  over  $K$  at  $v$  is defined as  $f_{X/K,v} = -\chi(X_{K_s}) + \chi(\mathcal{X}_{v,\bar{\kappa}_v}) + \delta_v$ , where  $\chi(X_{K_s})$ , respectively  $\chi(\mathcal{X}_{v,\bar{\kappa}_v})$  denotes the Euler-Poincaré characteristic of  $X_{K_s}$ , respectively  $\mathcal{X}_{v,\bar{\kappa}_v}$ . The term  $\delta_v$  denotes the Swan conductor of  $H^1(X_{\bar{K}}, \mathbb{Q}_\ell)$  at  $v$  (cf. [LiSa00, end of p. 414] for the definition, [Se69] for the definition of the Swan conductor, as well as [Bl87, §1]). Define  $f_{X/K} = \sum_v f_{X/K,v} \cdot \deg v$  to be the global conductor of the curve  $X/K$ .

The following result is a consequence of the subsequent lemma in [Bl87].

**Proposition 7.** *We have the inequality  $f_{J/K} \leq f_{X/K}$ .*

**Lemma 8.** [Bl87, Lemma 1.2] *Fix a place  $v$  of  $K$  and let  $I_v$  be an inertia subgroup of  $\text{Gal}(K_s/K)$  at  $v$ . Then :*

- (1)  $H_{\text{ét}}^i(X_{K_s}, \mathbb{Q}_\ell)^{I_v} \cong H_{\text{ét}}^i(\mathcal{X}_{v, \bar{\kappa}_v}, \mathbb{Q}_\ell)$  for  $i = 0, 1$ .
- (2) Let  $M_v$  be the free abelian group generated by the irreducible components of  $\mathcal{X}_{v, \bar{\kappa}_v}$ . Since the individual components are not necessarily defined over  $\kappa_v$ , there is an action of  $\hat{\mathbb{Z}} \cong \text{Gal}(\bar{\kappa}_v/\kappa_v)$  on  $M_v$ . Moreover, there is an exact sequence of  $\hat{\mathbb{Z}}$ -modules :

$$0 \rightarrow \mathbb{Q}_\ell(-1) \rightarrow M_v \otimes \mathbb{Q}_\ell(-1) \rightarrow H_{\text{ét}}^2(\mathcal{X}_{v, \bar{\kappa}_v}, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^2(X_{K_s}, \mathbb{Q}_\ell)^{I_v} \rightarrow 0.$$

**Remark 9.** The definition of the conductor given in [LiSa00] agrees with that given in [Bl87] (up to sign).

*Proof of Proposition 7.* It follows from the definition of  $f_{X/K, v}$ , Lemma 8 and the fact that the action of the Galois group  $\text{Gal}(K_s/K)$  on the étale cohomology groups  $H_{\text{ét}}^i(X_{K_s}, \mathbb{Q}_\ell)$  (for  $i = 0, 2$ ) is trivial that we have an equality :

$$f_{X/K, v} = \dim(H_{\text{ét}}^1(X_{K_s}, \mathbb{Q}_\ell)) - \dim(H_{\text{ét}}^1(X_{K_s}, \mathbb{Q}_\ell)^{I_v}) + m_v - 1 + \delta_v,$$

where  $m_v$  denotes the number of the irreducible components of  $\mathcal{X}_{v, \bar{\kappa}_v}$ . The proposition now follows from observing that  $H_{\text{ét}}^1(X_{K_s}, \mathbb{Q}_\ell) \cong H_{\text{ét}}^1(J_{K_s}, \mathbb{Q}_\ell)$  (cf. [Mi85, Corollary 9.6]).  $\square$

**Theorem 10.** *Let  $k$  be a finite field of characteristic  $p$ ,  $\mathcal{C}$  a smooth, projective, geometrically connected curve defined over  $k$  of genus  $g$  and denote by  $K = k(\mathcal{C})$  its function field. Let  $X/K$  be a smooth, projective, geometrically connected curve defined over  $K$  of genus  $d \geq 2$ . We assume that  $X$  is not defined over  $K^p$ . Then the following inequality holds :*

$$\#X(K) \leq p^{2d(2g+1)+f_{X/K}} \cdot 3^d \cdot (8d-2) \cdot d!.$$

*Proof.* Denote by  $X(K) = \{x_1, \dots, x_m\}$  the finite set of  $K$ -rational points of  $X$ . Let  $\Gamma$  be the subgroup of  $J(K)$  generated by the images  $\{j(x_1), \dots, j(x_m)\}$  of these points under the embedding  $j : X \hookrightarrow J$  of  $X$  into its Jacobian variety  $J$ . Observe that  $\#(\Gamma/p\Gamma) \leq \#(J(K)/pJ(K)) \leq p^{r+d} \leq p^{d(4g+1)+f_{X/K}}$  by Remark 2, Remark 3 and Proposition 7. The result is now a consequence of Theorem 1.  $\square$

**Remark 11.** We would now like to compare our result with a result similar in nature when we replace the one variable function field  $K$  defined over a finite field  $k$  by a number field  $K$ . In order to do this we refer to the work of Rémond (cf. [Re10]).

**Theorem 12** (Rémond). *Let  $X$  be a smooth, projective, geometrically connected curve of genus  $d \geq 2$  defined over a number field  $K$ , then one has*

$$\#X(K) \leq (2^{38+2d} \cdot [K : \mathbb{Q}] \cdot d \cdot \max(1, h_\theta))^{(r+1) \cdot d^{20}},$$

where  $h_\theta$  is the theta height of  $J$  and  $r = \text{rk}(J(K))$ .

**Remark 13.** Using Proposition 5.1 page 775 of [Re10], one has  $r \ll \log(f_{J/K})$ , as in the function field case, but the bound on the number of points is still depending on the height of the Jacobian variety. To be more precise, Rémond shows in *loc. cit.* how to produce a bound depending on the height of a model of the curve (and not of its Jacobian variety), but it seems difficult to get rid of this height. It would be a consequence of a conjecture of Lang and Silverman, as explained in

the introduction of [Pa12]. Note that in the function field case, the height of the variety  $J$  is comparable to the degree of the conductor  $f_{J/K}$ , as shown in [HiPa11, Corollary 6.12].

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